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Short Papers

Optimization of an Electrodynamic Basis for Determination of the Resonant Frequencies of Microwave Cavities Partially Filled with a Dielectric

JERZY KRUPKA

Abstract —**In this paper, a method of optimization of an electrodynamic basis is presented for determination of resonant frequencies of the microwave cavities containing dielectric samples. It is shown that the use of the suitable basis, consisting of several functions only, ensures a high accuracy of calculation of these frequencies.**

The presented method is useful for solving the boundary problem for the elliptic partial differential equation if the considered region has a regular boundary and is filled with inhomogeneous medium.

I. Theory

It is often necessary in practice to determine frequencies of the microwave resonant cavity in relation to the permittivity of the sample which fills this cavity. As it is known, this problem can be reduced to determination of the eigenvalues of the following boundary problem:

$$\begin{cases} L\phi = j\omega M\phi \\ \vec{n}\times\vec{E} = 0 \text{ on } S \end{cases} \tag{1}$$

where

$$L=\begin{bmatrix} 0 & \vec{\nabla}x \\ \vec{\nabla}x & 0 \end{bmatrix},\quad M=\begin{bmatrix} \epsilon_0\hat{\epsilon}_r & 0 \\ 0 & -\mu_0 \end{bmatrix},\quad \phi=\begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix}$$

$\hat{\epsilon}_r$ is the relative complex permittivity inside the cavity, $\vec{E}, \vec{H}$ are the electric and magnetic fields inside the cavity, and S is the surface of the cavity.

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Eigenvalues ω of this problem can be accurately calculated if the sample fills completely two of the cavity dimensions. In other cases, approximation methods must be used. In the most accurate of them the electromagnetic field is expanded into a series

$$\phi=\sum_i \alpha_i\phi_i \tag{2}$$

where $\{\alpha_i\}$ is the set of coefficients to be determined and $\{\phi_i\}$ is the set of basis functions (the electrodynamic basis). If the electrodynamic basis is given, the well-known methods (e.g., the Rayleigh–Ritz or the Galerkin methods [1], [2]) are employed to calculate eigenvalues ω and eigenvectors $\{\alpha_i\}$. The main problem is how to find the best electrodynamic basis. Usually the basis contains functions which are solutions of the boundary problem (1) for the empty cavity. In this paper, the basis is formed by functions which are solutions of the boundary problem (1) for the cavity partially filled with a dielectric in a suitable manner. The dielectric fills completely two cavity dimensions. The cavity with such a filling is called the basis cavity.

The nature of such modification can be explained as follows. We want to achieve the best similarity of distributions of electromagnetic fields in the basis cavity (Fig. 1(b)) and in the cavity which fields we are looking for (Fig. 1(a)). In this particular case, shown in Fig. 1(b), we can achieve that by changing ϵ_b and (or) the radius r_b.

Similar modification of an electrodynamic basis was presented for the first time in [4] for the rectangular cavity with a rectangular dielectric sample where the authors assumed that $\epsilon_b = \mathrm{Re}(\hat{\epsilon}_r)$ = const. In this paper, generalizations are made by assuming any ϵ_b value and by optimization of the choice of particular basis

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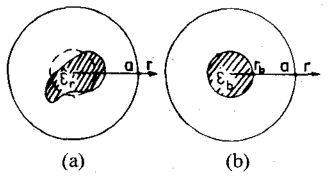

Fig. 1. Resonant cavities including dielectric samples. (a) The cavity whose solutions (resonant frequencies) are to be found. (b) The basis cavity with known solutions.

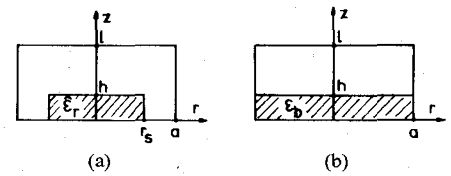

Fig. 2. Cylindrical TE_{011}-mode cavities including dielectric samples. (a) The cavity whose solutions (resonant frequencies) are to be found. (b) The basis cavity with known solutions.

functions. All calculations and experiments presented in this paper refer to the cylindrical quasi TE_{011}-mode cavity filled with a dielectric as it is shown in Fig. 2(a). The calculations are carried out by the Rayleigh–Ritz method. Since we are looking for frequency of the specific type of mode, the basis can be reduced to the class of rotational functions corresponding to quasi TE_{omn} modes of the basis cavity shown in Fig. 2(b).

The optimization of an electrodynamic basis consists in a choice of N basis functions from their infinite set to achieve the most accurate solution for fixing N and sample and cavity parameters. For the TE_{011} mode, the optimization is easy because its angular frequency value calculated by the Rayleigh–Ritz method is in excess of the accurate value. It is well known that Fourier's coefficients for higher modes of basis functions decrease with their frequency. Therefore, reducing somewhat the general problem one can find N of the most significant basis functions among K functions having the lowest frequency values. The choice of N elements among K can be made in $K!/(N-K)!/N!$ ways. Therefore, optimal basis functions were selected differently. First, the effect of particular basis functions on the convergence of the angular frequency of TE_{011} mode was examined. In order to do that, two basis functions were selected: the predominant TE_{011} and the examined TE_{omn}. The TE_{011}-mode frequency was then found by the Rayleigh–Ritz method using only these two modes. This was done for each of $K-1$ examined functions. Next, the best $N-1$ basis functions were chosen (those for which the lowest values of TE_{011}-mode frequency have been obtained). These functions together with the TE_{011} basis function made the best basis consisting of N functions, and were employed as in (2). The solution with all N functions were then obtained by the Rayleigh–Ritz method. Exemplary calculations have been carried out for the following cavity and sample parameters:

$$a = 25 \text{ mm}, l = 25 \text{ mm}, r_s = 12.5 \text{ mm}, h = 2 \text{ mm},$$
$$\mathrm{Re}(\hat{\epsilon}_r) = 10, \epsilon_b = 1,5, \text{ and } 10, K = 37, N \leqslant 15.$$

Fig. 3(a) represents the set of K basis functions from which optimized sets of basis functions were chosen. Fig. 3(c), (d), and (e) represent optimized sets containing $N = 15$ basis functions against a background of this set for various ϵ_b values. Numbers in these figures indicate the sequence of selecting of basis functions. Number 1 indicates the predominant TE_{011} function, number 2 indicates the best function among $K-1$ examined functions (in the meaning as it is described above), and so on.

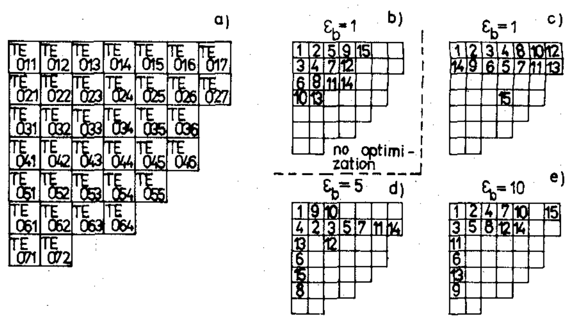

Fig. 3. (a) The set of $K = 37$ basis functions from which the sets of optimized functions are chosen. (b) The sequence of selecting of basis functions for the nonoptimized basis and $\epsilon_b = 1$ (functions are ordered in sequence of increasing frequencies). (c), (d), and (e) The sequence of selecting of basis functions for optimized bases for different ϵ_b values.

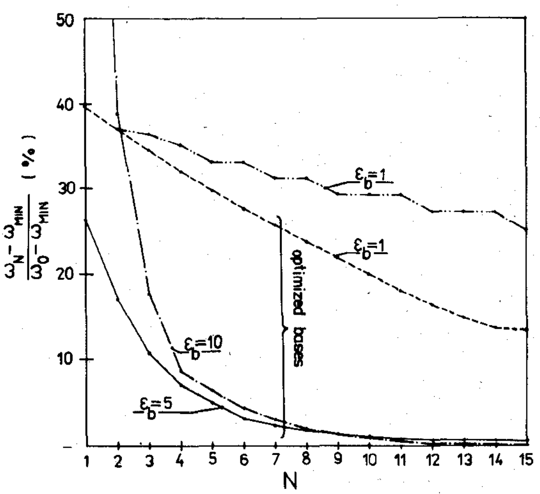

Fig. 4. Errors of frequency shift determination of cylindrical TE_{011}-mode cavity containing a dielectric sample for various sets of basis functions where ω_N is the angular frequency value calculated with the basis containing N functions, ω_0 is the angular frequency of an empty cavity, and ω_{MIN} is the lowest value of the angular frequency selected from values calculated with different 15-function bases.

From Fig. 3(c),(d),(e) one can also find the sets with $N < 15$ functions. For example, the set with $N = 4$ functions for $\epsilon_b = 5$ (Fig. 3(d)) contains functions marked in this figure numbers 1,2,3,4. This means (Fig. 3(a)) that these are TE_{011}, TE_{022}, TE_{023}, and TE_{025} modes.

Fig. 3(b) represents the set of nonoptimized basis functions in sequence of increasing frequencies for $\epsilon_b = 1$. Results of applications of different basis are represented in Fig. 4. This figure shows how far the suitable choice of the ϵ_b value effects a convergence and an accuracy of the solution. A particular case occurs when the basis contains only one function. Then the angular frequency value of the cavity can be obtained from the formula (3) (it is the first term of the Rayleigh–Ritz method). This formula may be also treated as the well-known Slater's theorem [5] assuming that the unperturbed cavity is the basis cavity

$$\frac{\hat{\omega}_1^2 - \omega_b^2}{\hat{\omega}_1^2} = -\frac{\iiint_{V_c} \epsilon_0(\hat{\epsilon}_r - \epsilon_b)\vec{E}_b\vec{E}_b^* \, dv}{\iiint_{V_c} \epsilon_0\epsilon_b\vec{E}_b\vec{E}_b^* \, dv} \tag{3}$$

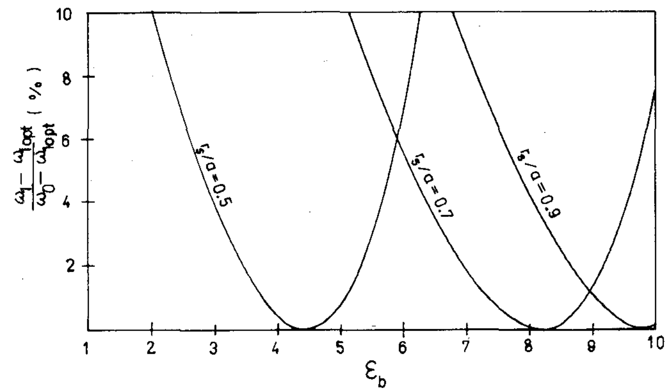

Fig. 5. Errors of frequency shift determination of cylindrical TE_{011}-mode cavity containing a dielectric sample plotted against ϵ_b. Calculations have been carried out with single-function bases.

where $\hat{\omega}_1$ is the complex angular frequency of the cavity calculated with the single function basis (if losses do exist ω_1 is real), ω_b is the angular frequency of the basis cavity, $\vec{E}_b$ is the electric field inside the basis cavity, and V_c is the volume of the cavity.

For $\epsilon_b = 1$, the formula (3) yields a well-known zero approximation perturbation method. It is clear that there exists such $\epsilon_b = \epsilon_{bopt}$ for which the lowest angular frequency value $\omega_1 = \omega_{1opt}$ is obtained. Obviously, the ϵ_{bopt} value depends on the sample radius, e.g., $\epsilon_{bopt} \to \mathrm{Re}(\hat{\epsilon}_r)$ for $r_s/a \to 1$ and $\epsilon_{bopt} \to 1$ for $r_s/a \to 0$. Fig. 5 represents standardized differences between angular frequencies ω_1 for any ϵ_b values and the angular frequency ω_{1opt}. ω_1 values can be treated as the starting points of the Rayleigh–Ritz method for different bases (addition of the subsequence basis functions in the Rayleigh–Ritz method improves the previous result).

Let us now return to the results shown in Fig. 4. Two characteristic ranges of ϵ_b values can be distinguished.

1) $\epsilon_b \geqslant \epsilon_{bopt}$. Then if ϵ_b values increase, ω_1 values (starting points) increase and the convergence of the solution becomes better. For a small number of basis functions, the first element has a decisive significance so the best solution is obtained for the basis with $\epsilon_b = \epsilon_{bopt}$. For a larger number of basis functions ($N > 10$) both elements have similar significance so solutions obtained for bases with different ϵ_b values have similar accuracy.

2) $\epsilon_b < \epsilon_{bopt}$. Then if the ϵ_b values decrease, ω_1 values increase and the convergence of the solution becomes worse. Therefore the application of the basis consisting of empty cavity modes is not efficient.

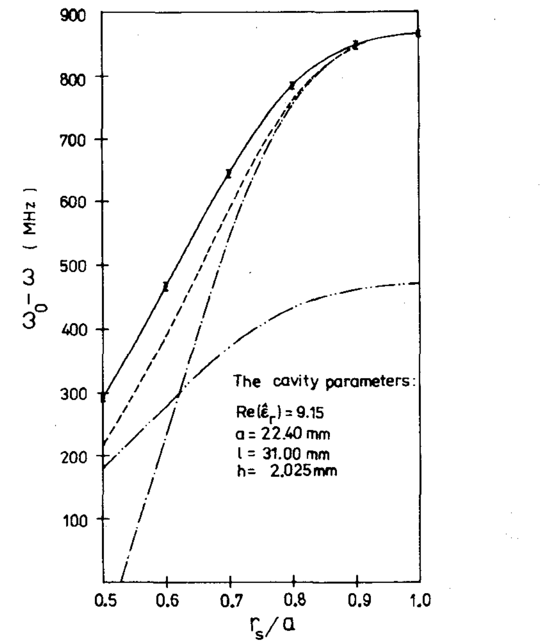

Fig. 6. Angular frequency shifts of the cylindrical TE_{011}-mode cavity with Al_2O_3 sample plotted against the sample diameter. Theoretical results are compared with experimental results. ——Rayleigh–Ritz method with the optimal 15-function basis and $\epsilon_b = \epsilon_{bopt}$. –·– Rayleigh–Ritz method with the single-function basis and $\epsilon_b = \mathrm{Re}(\hat{\epsilon}_r)$. –··– Rayleigh–Ritz method with the single-function basis and $\epsilon_b = 1$ (zero approximation perturbation method). ------ Rayleigh–Ritz method with the optimal single-function basis ($\epsilon_b = \epsilon_{bopt}$). Φ Experimental points.

TABLE I

RESONANT FREQUENCIES OF THE CYLINDRICAL TE_{011}-MODE CAVITY FOR DIFFERENT DIAMETERS OF THE SAMPLE WHICH FILLS THAT CAVITY

No	r_s mm	r_s/a	f MHz measured	$\omega = 2\pi f$ MHz measured	ω MHz calculated with 15-function optimized basis	ε_{bopt} approximate values	Comment
1	0	0	9535.50	59913.3	——	1.00 (exact)	empty cavity
2	22.3	1.0	9400.40	59064.5	——	9.15 (exact)	basis cavity $\varepsilon_{bopt} = \mathrm{Re}(\hat{\varepsilon}_r)$
3	20.2	0.9	9402.50	59077.7	59078.0	9.00	partially filled cavity
4	17.9	0.8	9413.20	59144.9	59146.6	8.75	
5	15.7	0.7	9435.00	59281.9	59282.4	7.75	
6	13.4	0.6	9463.40	59460.3	59462.7	6.25	
7	11.2	0.5	9492.10	59640.6	59644.1	4.00	

II. EXPERIMENTS

Experiments have been carried out, first of all, in order to check whether there exist differences between angular frequency values obtained by the method described above and experimental values which are accurate. In order to do that, first the value of the permittivity $\mathrm{Re}(\hat{\epsilon}_r)$ of the Al_2O_3 sample filling completely the cross section of the cavity was calculated. The $\mathrm{Re}(\hat{\epsilon}_r)$ value was found from the two first measurements represented in Table I. Next, the sample diameter was gradually reduced and the frequency of the cavity with the sample was measured each time. For the $\mathrm{Re}(\hat{\epsilon}_r)$ value obtained by experiments, theoretical calculations were carried out by the Rayleigh–Ritz method. Results are shown in Fig. 6 and in Table I.

One can see that there are insignificant differences between experimental results and those obtained by the Rayleigh–Ritz method using 15-function optimized basis. But there are significant differences between experimental results and those obtained by various perturbation formulas (the Rayleigh–Ritz method with single-function bases).

III. CONCLUSIONS

By optimization of an electrodynamic basis presented in this paper, the high calculation accuracy of frequencies of the microwave cavity with a dielectric sample is obtained, using several functions only. Here, however, some difficulties arise. First, one must solve K transcendental equations in order to calculate angular frequencies of the basis cavity. Second, the new basis functions are more complicated than empty cavity basis functions. Therefore such an optimization is recommended in the following cases: 1) when a very high accuracy of calculations is required, and 2) when the computation time must be short or/and the available computer memory is limited.

If required computation accuracy is not too high, the variant of the perturbation method described in this paper is recommended rather than the Rayleigh–Ritz method with empty cavity basis functions. Although this paper refers to the specific boundary problem, conclusions are general and can be useful for other similar cases.

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Propagation in Longitudinally Magnetized Compressible Plasma Between Two Parallel Planes

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Abstract—**The propagation of plasma waves in compressible, single fluid, macroscopic plasma, between two parallel, perfectly conducting planes, with longitudinal magnetostatic field parallel to the boundaries and in the direction of propagation is investigated for the different hybrid plasma wave modes of propagation.**

I. Propagation in Parallel Plane Waveguide

The propagation of plasma waves in compressible, single fluid, macroscopic plasma, between two parallel perfectly conducting planes, with transverse magnetostatic field parallel to the boundaries, has been recently investigated [1]. In the present short paper the theory will be extended to the case where the magnetostatic field is parallel to the boundaries and in the longitudinal direction of propagation of the waves.

Using small signal theory approximation, and assuming harmonic time variation $e^{+i\omega t}$, the wave equation for the electric field $\bar{E}$ in the magnetoplasma has been found [1] in the form

$$-\frac{1}{k_0^2}\nabla\times\nabla\times\bar{E}+\frac{1}{k_1^2}\nabla(\nabla\cdot\bar{E})+(1-X)\bar{E}+\frac{1}{k_0^2}(\nabla\times\nabla\times\bar{E}-k_0^2\bar{E})\times i\bar{Y}=0 \quad (1)$$

where k_0 is the electromagnetic wave number, k_1 is the acoustic wave number, X is proportional to the average plasma density N_0, and $\bar{Y}$ is proportional to the magnetostatic field $\bar{H}_0$. The wave magnetic field $\bar{H}$ and the wave velocity field $\bar{u}$ may be found [1] from the plasma wave electric field $\bar{E}$.

It is assumed that the compressible plasma is confined by two perfectly conducting parallel planes at $x=0$ and $x=a$, with the magnetostatic field in the longitudinal direction of propagation z

$$\bar{Y}=Y\hat{z}=\frac{e\mu H_0}{m\omega}\hat{z}. \quad (2)$$

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Since the solution will be independent of the y-axis, one may assume that each one of the plasma wave components will be in the form

$$E_j(x,z)=E^j(\alpha)e^{i\alpha x}e^{i(\omega t-\gamma z)}, \qquad j=x,y,z. \quad (3)$$

The constant γ represents the propagation constant of the plasma wave modes propagating in the z direction, and it depends on α to be determined from the boundary conditions.

Substituting (2) and (3) in (1), and taking from (3) $\partial/\partial x = i\alpha$, $\partial/\partial y=0$, $\partial/\partial z=-i\gamma$, one obtains three homogeneous linear algebraic equations for E^x, E^y, and E^z. For a nontrivial solution, the determinant of the coefficients should be zero, and developing this determinant, one obtains

$$\begin{aligned}&\left[k_0^2(1-X)-(\alpha^2+\gamma^2)\right]^2\left[k_0^2(1-X)-\delta(\alpha^2+\gamma^2)\right]\\&+Y^2(k_0^2-\alpha^2-\gamma^2)\left[k_0^2X(k_0^2-\gamma^2)-(k_0^2-\delta\gamma^2)(k_0^2-\alpha^2-\gamma^2)\right]=0\end{aligned} \quad (4)$$

where

$$k_0^2=\omega^2\mu\epsilon$$

and

$$\delta=k_0^2/k_1^2.$$

Equation (4) could be rearranged to give a cubic equation in terms of α^2, with the coefficients of the equation depending on γ^2.

According to the theory of linear algebraic equations, one may express E^x and E^y in terms of E^z. All the other plasma wave components $\bar{H}$ and $\bar{u}$ of the plasma wave hybrid modes could be expressed in terms of E^z as well, by using the relationships given previously [1]. The following boundary conditions will be applied in the present problem:

$$E_z=0 \qquad \text{at } x=0 \text{ and } x=a \quad (5a)$$

$$E_y=0 \qquad \text{at } x=0 \text{ and } x=a \quad (5b)$$

$$u_x=0 \qquad \text{at } x=0 \text{ and } x=a. \quad (5c)$$

II. The Plasma Waves Hybrid Modes

The equation which relates α^2 with the propagation constant γ of the plasma waves hybrid modes is given in (4). For a given γ, one may solve the cubic equation (4) in order to obtain the corresponding characteristic values $\pm\alpha_1$, $\pm\alpha_2$, and $\pm\alpha_3$ in terms of γ. It may be assumed, therefore, that the longitudinal electric field component E_z of the plasma waves hybrid mode is given in the form

$$E_z=\left[A_1\sin\alpha_1x+B_1\cos\alpha_1x+A_2\sin\alpha_2x+B_2\cos\alpha_2x+A_3\sin\alpha_3x+B_3\cos\alpha_3x\right]e^{i(\omega t-\gamma z)} \quad (6)$$

where A_1, A_2, A_3 and B_1, B_2, B_3 are arbitrary constants. Using (6) and the analysis described above, one may find E_y in terms of the trigonometric functions and the arbitrary constants in (6) and the constants $D_m=D_m(\alpha_m^2,\gamma)$, where $m=1,2,3$. Using (6) and the corresponding relationship in the previous paper [1], one may find u_x in terms of the trigonometric functions and the arbitrary constants in (6) and the constants $P_m(\alpha_m^2,\gamma)$, where $m=1,2,3$.

Using (6) in the boundary conditions (5a) one obtains

$$B_1+B_2+B_3=0 \quad (7a)$$

$$A_1\sin\alpha_1a+B_1\cos\alpha_1a+A_2\sin\alpha_2a+B_2\cos\alpha_2a+A_3\sin\alpha_3a+B_3\cos\alpha_3a=0. \quad (7b)$$